Lecture 10:

Main idea of numerical spectral method

Diff. eqt $\longrightarrow$ Algebraic eqt $\longrightarrow$ solution $\vec{u}$

(e.g. $\frac{d^2u}{dx^2} = f$)

DFT on $\vec{f}$ $\left(\frac{\overline{A_w}}{n}\vec{f}\right)$ — Fast?

Fast!

inverse DFT on $\hat{u}$ $\left(A_w \hat{u}\right)$ — Fast?

Remark:

- To develop an efficient numerical spectral method, we need to compute $A_w \hat{u}$ and $\frac{\overline{A_w}}{n}\vec{f}$ fast.
- Computational cost for $\underset{(n\times n)}{A_w}\hat{u}$ is $O(n^2)$.

Goal: Reduce the computational cost to $O(n\log n)$

e.g. $n = 2^{10}$, $n^2 = 2^{20}$, $n\log n = 10\cdot 2^{10} < 2^{14}$. $\therefore\ 2^6 = 64$ times faster!

Fast Fourier Transform (FFT) (Colley and Tukey, 1965)

Let $F_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \cdots & \omega_n^{n-1} \\ \vdots & \vdots & & \\ 1 & \omega_n^{n-1} & \cdots & \omega_n^{(n-1)^2} \end{pmatrix}$ where $\omega_n = e^{i\left(\frac{2\pi}{n}\right)}$.

Let $\vec{y} = F_n \vec{x}$, where $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$. Suppose $n = 2m$.

Then, for each $0 \le j \le n-1$,

$$y_j = \sum_{k=0}^{n-1} \omega_n^{jk} x_k = \sum_{k=0}^{2m-1} \omega_{2m}^{kj} x_k .$$

Divide $k = 0, 1, 2, \ldots, 2m-1$ into two parts:

<u>Part 1:</u> $0, 2, 4, 6, \ldots, 2(m-1)$ (Even)

<u>Part 2:</u> $1, 3, 5, 7, \ldots, 2m-1$ (Odd)

Then: $$y_j = \underbrace{\sum_{k=0}^{m-1} \omega_n^{2kj} x_{2k}}_{\text{Part 1}} + \underbrace{\sum_{k=0}^{m-1} \omega_n^{(2k+1)j} x_{2k+1}}_{\text{Part 2}}$$

$$= \sum_{k=0}^{m-1} \omega_m^{kj} \overset{(\vec{x}')_k}{x_{2k}} + \omega_n^{j} \sum_{k=0}^{m-1} \omega_m^{kj} \overset{(\vec{x}'')_k}{x_{2k+1}}$$

$$\left(\because \ \omega_{2m}^{2k} = e^{i\left(\frac{2\pi}{2m}\right)2k} = e^{i\left(\frac{2\pi}{m}\right)k} = \omega_m^k \right)$$

Denote $\vec{x}' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2m-2} \end{pmatrix}$, $\vec{x}'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$. Let $\vec{y}' = F_m \vec{x}'$ and $\vec{y}'' = F_m \vec{x}''$.

$$\therefore \ y_j = (\overbrace{F_m}^{m \times m} \vec{x}')_j + \omega_n^j (\overbrace{F_m}^{m \times m} \vec{x}'')_j \quad \text{for } j = 0, 1, 2, \ldots, m-1$$

$$= \underbrace{(\vec{y}')_j}_{\text{j-th entry of } \vec{y}'} + \omega_n^j \underbrace{(\vec{y}'')_j}_{\text{j-th entry of } \vec{y}''}$$

$$y_{j+m} = \sum_{k=0}^{m-1} \omega_n^{2k(j+m)} x_{2k} + \sum_{k=0}^{m-1} \omega_n^{(2k+1)(j+m)} x_{2k+1} \quad \text{for } j = 0, 1, 2, \ldots, m-1$$

Capture $y_m, y_{m+1}, \ldots, y_{2m-1}$

$$= \sum_{k=0}^{m-1} \omega_m^{kj}\, \omega_m^{km} (\vec{x}\,')_k + \sum_{k=0}^{m-1} \omega_m^{k(j+m)}\, \omega_n^{j+m} (\vec{x}\,'')_k$$

$$\omega_m^{km} = e^{i\left(\frac{2\pi}{m}\right)km} = 1$$

$$\omega_m^{k(j+m)} = \omega_m^{kj}\, \omega_m^{km} = \omega_m^{kj}$$

$$\omega_n^{j+m} = \omega_n^{j}\, \omega_n^{m} = \omega_n^{j}\, e^{i\left(\frac{2\pi}{2m}\right)\cdot m} = -\omega_n^{j}$$

$$\therefore \quad y_{j+m} = \sum_{k=0}^{m-1} \omega_m^{kj} (\vec{x}\,')_k - \omega_n^{j} \sum_{k=0}^{m-1} \omega_m^{kj} (\vec{x}\,'')_k$$

$$y_{j+m} = (\underbrace{F_m}_{m\times m} \vec{x}\,')_j - \omega_n^{j} (\underbrace{F_m}_{m\times m} \vec{x}\,'')_j \quad \text{for } j = 0, 1, 2, \ldots, m-1$$

__Note__: $n \times n$ matrix multiplication becomes $\frac{n}{2} \times \frac{n}{2} = m \times m$ matrix multiplication.

For simplicity, we denote:
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \\ \vdots \\ v_n w_n \end{pmatrix}$$

Then:
$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} = \vec{y}\,' + \begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix} \otimes \vec{y}\,'' \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}\,' - \begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix} \otimes \vec{y}\,''$$

Summary of FFT

Step 1: Split $\vec{x}$ into $\vec{x}\,' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2(m-1)} \end{pmatrix}$ and $\vec{x}\,'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$

Step 2: Compute $\vec{y}\,' = F_m \vec{x}\,'$ and $\vec{y}\,'' = F_m \vec{x}\,''$, where $F_m = \underbrace{\tfrac{n}{2} \times \tfrac{n}{2}}_{}$ matrix ($= m \times m$)

$O(m^2)$ $\qquad$ $O(m^2)$

Step 3: Compute:
$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} = \vec{y}\,' + \overbrace{\begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix}}^{\vec{\omega}_m} \otimes \vec{y}\,'' \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}\,' - \overbrace{\begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix}}^{\vec{\omega}_m} \otimes \vec{y}\,''$$

$O(m)$ $\qquad$ $O(m)$

Remark: Computational cost: $\mathcal{O}(m^2) + \mathcal{O}(m)$ (multiplication + addition)

$$\approx$$

$$\mathcal{O}(m^2)$$